Absolutely summing linear operators into spaces with no finite cotype

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Abstract

Given an infinite-dimensional Banach space X and a Banach space Y with no finite cotype, we determine whether or not every continuous linear operator from X to Y is absolutely $(q;p)$-summing for almost all choices of p and q, including the case $p=q$. If X assumes its cotype, the problem is solved for all choices of p and q. Applications to the theory of dominated multilinear mappings are also provided.

Introduction

Given Banach spaces X and Y, the question of whether or not every continuous linear operator from X to Y is absolutely $(q;p)$-summing has been the subject of several classical works, such as Bennet [2], Carl [6], Dubinsky, Pełczyński and Rosenthal [8], Garling [9], Kwapień [11], Lindenstrauss and Pełczyński [12] and many others. In this note we address this question for range spaces Y having no finite cotype (such spaces are abundant in Banach space theory). For arbitrary domain spaces X the results we prove settle the question for almost every choice of p and q (Theorem 2.3), including the case $p=q$ (Corollary 2.2). For domain spaces X having cotype $\inf\{q : X \text{ has cotype } q\}$ (by far most Banach spaces enjoy this property) our results settle the question for all choices of p and q (Corollary 2.4). Applications of these results to the theory of dominated multilinear mappings are given in a final section.

1 Background and notation

Throughout this note, n will be a positive integer, $X, X_1, ..., X_n$ and Y will represent Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The symbol X' represents the topological dual of X and B_X the closed unit ball of X. The Banach space of all continuous linear operators from X to Y, endowed with the usual sup norm, will be denoted by $\mathcal{L}(X;Y)$.

*Supported by CNPq Project 202162/2006-0.
†Supported by CNPq Projects 471054/2006-2 and 308084/2006-3.
2000 Mathematics Subject Classification. Primary 47B10; Secondary 46G25.

Given $1 \leq p < +\infty$ and a Banach space X, the linear space of all sequences $(x_j)_{j=1}^{\infty}$ in X such that $\|(x_j)_{j=1}^{\infty}\|_p := (\sum_{j=1}^{\infty} \|x_j\|^p)^{\frac{1}{p}} < \infty$ will be denoted by $\ell_p(X)$. By $\ell_p^w(X)$ we represent the linear space composed by the sequences $(x_j)_{j=1}^{\infty}$ in X such that $(\varphi(x_j))_{j=1}^{\infty} \in \ell_p$ for every $\varphi \in X'$. A norm $\|\cdot\|_{w,p}$ on $\ell_p^w(X)$ is defined by $\|(x_j)_{j=1}^{\infty}\|_{w,p} := \sup_{\varphi \in B_{X'}} (\sum_{j=1}^{\infty} |\varphi(x_j)|^p)^{\frac{1}{p}}$. A linear operator $u \colon X \longrightarrow Y$ is said to be absolutely (q,p)-summing (or simply (q,p)-summing), $1 \leq p \leq q < +\infty$, if $(u(x_j))_{j=1}^{\infty} \in \ell_q(Y)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$. By $\Pi_{q;p}(X;Y)$ we denote the subspace of $\mathcal{L}(X;Y)$ of all absolutely (q,p)-summing operators, which becomes a Banach space with the norm $\pi_{q;p}(u) := \sup\{\|(u(x_j))_{j=1}^{\infty}\|_q : (x_j)_{j=1}^{\infty} \in B_{\ell_p^w(X)}\}$. If $p = q$ we simply say that u is absolutely p-summing (or p-summing) and simply write $\Pi_p(X;Y)$ for the corresponding space.

Given a Banach space X, we put $r_X := \inf\{q : X \text{ has cotype } q\}$. Clearly $2 \leq r_X \leq +\infty$.

For $1 \leq p < +\infty$, p^* denotes its conjugate index, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$ ($p^* = 1$ if $p = +\infty$).

For the theory of absolutely summing operators and for any unexplained concepts we refer to Diestel, Jarchow and Tonge [7].

2 Main results

Henceforth p, q and r will be real numbers with $1 \leq p \leq q < +\infty$ and $1 \leq r \leq +\infty$.

Theorem 2.1. *Let Y be a Banach space with no finite cotype and suppose that ℓ_r is finitely representable in X. Then there exists a continuous linear operator from X to Y which fails to be $(q;p)$-summing if either $1 \leq q < r$ or $p \geq r^*$.*

Proof. Assume first that $r < +\infty$. By $(e_j)_{j=1}^{\infty}$ we mean the canonical unit vectors of ℓ_r. If $1 \leq q < r$, then $\left(\frac{e_j}{j^{\frac{1}{q}}}\right)_{j=1}^{\infty} \in \ell_1^w(\ell_r) \subseteq \ell_p^w(\ell_r)$ because $q < r$ and $\left(\frac{e_j}{j^{\frac{1}{q}}}\right)_{j=1}^{\infty} \notin \ell_q(\ell_\infty)$ (obvious). Moreover, for every $n \in \mathbb{N}$,

$$\sup_n \left\| \left(\frac{e_j}{j^{\frac{1}{q}}}\right)_{j=1}^{n} \right\|_{\ell_p^w(\ell_r)} < +\infty \text{ and } \sup_n \left\| \left(\frac{e_j}{j^{\frac{1}{q}}}\right)_{j=1}^{n} \right\|_{\ell_q(\ell_\infty)} = +\infty.$$

So, for every positive integer n, if $u_n \colon \ell_r^n \longrightarrow \ell_\infty^n$ denotes the formal inclusion, then

$$\sup_n \pi_{q;p}(u_n) = +\infty \text{ and } \|u_n\| = 1.$$

The same is true if $p \geq r^*$ as $(e_j)_{j=1}^{\infty} \in \ell_{r^*}^w(\ell_r) \subset \ell_p^w(\ell_r)$ and $(e_j)_{j=1}^{\infty} \notin \ell_q(\ell_\infty)$.

We know that ℓ_∞ is finitely representable in Y from the celebrated Maurey-Pisier Theorem [1, Theorem 11.1.14 (ii)] and that ℓ_r is finitely representable in

X by assumption. So, for each $n \in \mathbb{N}$, there exist a subspace Y_n of Y, a subspace X_n of X and linear isomorphisms T and R

$$\ell_\infty^n \xrightarrow{T} Y_n \xrightarrow{T^{-1}} \ell_\infty^n \text{ and } \ell_r^n \xrightarrow{R} X_n \xrightarrow{R^{-1}} \ell_r^n$$

so that $\|T\| = \|R\| = 1$, $\left\|T^{-1}\right\| < 2$ and $\left\|R^{-1}\right\| < 2$. Now consider the chain

$$\ell_r^n \xrightarrow{R} X_n \xrightarrow{R^{-1}} \ell_r^n \xrightarrow{u_n} \ell_\infty^n \xrightarrow{T} Y_n \xrightarrow{T^{-1}} \ell_\infty^n.$$

Since $\|R\| = 1$, we conclude that

$$\pi_{q;p}(u_n) = \pi_{q;p}(u_n \circ R^{-1} \circ R) \leq \pi_{q;p}(u_n \circ R^{-1}) \|R\| = \pi_{q;p}(u_n \circ R^{-1}).$$

Hence the operator $u_n \circ R^{-1} \colon X_n \longrightarrow l_\infty^n$ is so that

$$\sup_n \pi_{q;p}(u_n \circ R^{-1}) = +\infty \text{ and } \sup_n \left\|u_n \circ R^{-1}\right\| < +\infty.$$

Since ℓ_∞^n is an injective Banach space, there is a norm preserving extension $v_n \colon X \longrightarrow \ell_\infty^n$ of $u_n \circ R^{-1}$. It is immediate that

$$\sup_n \pi_{q;p}(v_n) = +\infty \text{ and } \sup_n \|v_n\| < +\infty. \tag{1}$$

Consider now the operator $T \circ v_n \colon X \longrightarrow Y_n$. Since $\left\|T^{-1}\right\| < 2$, we have

$$\pi_{q;p}(v_n) = \pi_{q;p}(T^{-1} \circ T \circ v_n) < 2\pi_{q;p}(T \circ v_n). \tag{2}$$

From (1), (2) and $\|T\| = 1$ we get

$$\sup_n \pi_{q;p}(T \circ v_n) = \infty \text{ and } \sup_n \|T \circ v_n\| < +\infty. \tag{3}$$

By composing $T \circ v_n$ with the formal inclusion $i \colon Y_n \longrightarrow Y$ we obtain the operator $i \circ T \circ v_n \colon X \longrightarrow Y$. Combining the injectivity of $\Pi_{q;p}$ [7, Proposition 10.2] with (3) we have

$$\sup_n \|i \circ T \circ v_n\|_{as(q;p)} = \infty \text{ and } \sup_n \|i \circ T \circ v_n\| < \infty.$$

Calling on the Open Mapping Theorem we conclude that $\Pi_{q;p}(X, Y) \neq \mathcal{L}(X, Y)$.

Suppose now that ℓ_∞ is finitely representable in X. Since every Banach space is finitely representable in c_0, ℓ_r is finitely representable in c_0, hence in ℓ_∞, for every $1 \leq r < +\infty$. It follows that ℓ_r is finitely representable in X for every $1 \leq r < +\infty$, so the result holds for every $1 \leq r < +\infty$ by the first part of the proof, hence for $r = +\infty$. $\square$

Corollary 2.2. *Regardless of the infinite-dimensional Banach space X, the Banach space Y with no finite cotype and $p \geq 1$, there exists a continuous linear operator from X to Y which fails to be p-summing.*

Proof. By Maurey-Pisier Theorem [1, Theorem 11.3.14] we know that ℓ_{r_X} is finitely representable in X, so Theorem 2.1 provides a continuous linear operator $u \colon X \longrightarrow Y$ which fails to be p-summing for every $p \geq r_X^*$. Since $\Pi_r \subseteq \Pi_s$ if $r \leq s$ [7, Theorem 2.8], it follows that u fails to be p-summing for every $p \geq 1$. □

Next result settles the question $\Pi_{q;p}(X;Y) \overset{??}{=} \mathcal{L}(X;Y)$ for Y with no finite cotype for almost all choices of p and q:

Theorem 2.3. *Let Y be a Banach space with no finite cotype and X be an infinite-dimensional Banach space. Then:*
(a) $\Pi_{q;p}(X;Y) \neq \mathcal{L}(X;Y)$ if either $1 \leq q < r_X$ or $p \geq r_X^*$ or $1 < p < r_X^*$ and $q < \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$.
(b) $\Pi_{q;p}(X;Y) = \mathcal{L}(X;Y)$ if either $p = 1$ and $q > r_X$ or $1 < p < r_X^*$ and $q > \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$.

Proof. (a) Since ℓ_{r_X} is finitely representable in X (Maurey-Pisier Theorem), the case $1 \leq q < r_X$ and the case $p \geq r_X^*$ follow from Theorem 2.1. Suppose $1 < p < r_X^*$ and $q < \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$. From the previous cases we know that $\Pi_{s;r_X^*}(X;Y) \neq \mathcal{L}(X;Y)$ for every $s \geq 1$. So the proof will be complete if we show that $\Pi_{q;p}(X;Y) \subseteq \Pi_{s;r_X^*}(X;Y)$ for sufficiently large s. By [7, Theorem 10.4] it suffices to show that there exist a sufficiently large s so that $q \leq s$, $r_X^* \leq s$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{r_X^*} - \frac{1}{s}$. From

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{p} - \frac{1}{\frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}} = \frac{1}{p} - \left(\frac{1}{p} - \frac{1}{r_X^*}\right) = \frac{1}{r_X^*}$$

we can choose $s \geq \max\{q, r_X^*\}$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{r_X^*} - \frac{1}{s}$, completing the proof of (a).
(b) If $q > r_X$, then X has cotype q, hence the identity operator on X is $(q;1)$-summing, so $\Pi_{q;1}(X;Y) = \mathcal{L}(X;Y)$. Suppose $1 < p < r_X^*$ and $q > \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$. Calling on [7, Theorem 10.4] once again we have that $\Pi_{r_X+\varepsilon;1}(X;Y) \subset \Pi_{q;p}(X;Y)$ for a sufficiently small $\varepsilon > 0$. From the previous case we know that $\Pi_{r_X+\varepsilon;1}(X;Y) = \mathcal{L}(X;Y)$, so $\Pi_{q;p}(X;Y) = \mathcal{L}(X;Y)$ as well. □

The only cases left open are (i) $p = 1$ and $q = r_X$, (ii) $1 < p < r_X^*$ and $q = \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$. For spaces X having cotype r_X the problem is completely settled:

Corollary 2.4. *Suppose that Y has no finite cotype and that X is infinite-dimensional and has cotype r_X. Then $\Pi_{q;p}(X;Y) = \mathcal{L}(X;Y)$ if and only if either $p = 1$ and $q \geq r_X$ or $1 < p < r_X^*$ and $q \geq \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$.*

Proof. As mentioned above, by Theorem 2.3 it suffices to consider the cases (i) $p = 1$ and $q = r_X$, (ii) $1 < p < r_X^*$ and $q = \frac{1}{\frac{1}{p} - \frac{1}{r_X^*}}$. Since X has cotype r_X, the identity operator on X is $(r_X;1)$-summing, so (i) is done. By [7, Theorem 10.4]

we have that $\Pi_{r_X;1}(X;Y) \subset \Pi_{\frac{1}{\frac{1}{p}-\frac{1}{r_X^*}};p}(X;Y)$ whenever $1 < p < r_X^*$, so (ii) follows from (i). $\square$

Note that Corollary 2.4 improves the linear case of [15, Corollary 6].

The next consequence of Theorem 2.3, which is closely related to a classical result of Maurey-Pisier [14, Remarque 1.4] and to [5, Example 2.1], shows that fixed an infinite-dimensional Banach space X, the number $\inf\{q : \Pi_{q;1}(X;Y) = \mathcal{L}(X;Y)\}$ does not depend on the Banach space with no finite cotype Y.

Corollary 2.5. *Let X be an infinite-dimensional Banach space. Then $r_X = \inf\{q : \Pi_{q;1}(X;Y) = \mathcal{L}(X;Y)\}$ regardless of the Banach space Y with no finite cotype.*

3 Applications to the multilinear theory

One of the most interesting and most studied multilinear generalizations of the ideal of absolutely p-summing linear operators is the class of p-dominated multilinear mappings. A continuous n-linear mapping $A\colon X_1 \times \cdots \times X_n \longrightarrow Y$ is $(p_1, \ldots, p_n)$-dominated, $1 \le p_1, \ldots, p_n < +\infty$, if $(A(x_j^1, \ldots, x_j^n))_{j=1}^{\infty} \in \ell_q(Y)$ whenever $(x_j^k)_{j=1}^{\infty} \in \ell_{p_k}^w(X_k)$, $k = 1, \ldots, n$, where $\frac{1}{q} = \frac{1}{p_1} + \cdots + \frac{1}{p_n}$. If $p_1 = \cdots = p_n = p$ we simply say that A is p-dominated. For details we refer to [4, 13].

Continuous bilinear forms on either an $\mathcal{L}_\infty$-space, or the disc algebra $\mathcal{A}$ or the Hardy space H^∞ are 2-dominated [4, Proposition 2.1]. On the other hand, partially solving a problem posed in [4], in [10, Lemma 5.4] it was recently shown that for every $n \ge 3$, every infinite-dimensional Banach space X and any $p \ge 1$, there is a continuous n-linear form on X^n which fails to be p-dominated. As to vector-valued bilinear mappings, all that is known, as far as we know, is that for every $\mathcal{L}_\infty$-spaces X_1, X_2, every infinite-dimensional space Y and any $p \ge 1$, there is a continuous bilinear mapping $A : X_1 \times X_2 \to Y$ which fails to be p-dominated [3, Theorem 3.5]. Besides of giving an alternative proof of [10, Lemma 5.4], we fill in this gap concerning vector-valued bilinear mappings by generalizing [3, Theorem 3.5] to arbitrary infinite-dimensional spaces X_1, X_2, Y.

Proposition 3.1. *Let X_1, X_2 and Y be infinite-dimensional Banach spaces and let $p_1, p_2 \ge 1$. Then there exists a continuous bilinear mapping $A\colon X_1 \times X_2 \longrightarrow Y$ which fails to be (p_1, p_2)-dominated.*

Proof. Suppose, by contradiction, that every continuous bilinear mapping from $X_1 \times X_2$ to Y is (p_1, p_2)-dominated. A straightforward adaptation of the proof of [3, Lemma 3.4] gives that every continuous linear operator from X_1 to $\mathcal{L}(X_2;Y)$ is p_1-summing. From [7, Proposition 19.17] we know that $\mathcal{L}(X_2;Y)$ has no finite cotype, so Corollary 2.2 assures that there is a continuous linear operator from X_1 to $\mathcal{L}(X_2;Y)$ which fails to be p_1-summing. This contradiction completes the proof. $\square$

The same reasoning extends [10, Lemma 5.4] to $(p_1, \ldots, p_n)$-dominated n-linear mappings (for eventually different $p_1, \ldots, p_n$) on $X_1 \times \cdots \times X_n$ (for eventually different spaces $X_1, \ldots, X_n$):

Proposition 3.2. *Let $n \geq 3$, $X_1, \ldots, X_n$ be Banach spaces at least three of them infinite-dimensional and let $p_1, \ldots, p_n \geq 1$. Then there exists a continuous n-linear form $A\colon X_1 \times \cdots \times X_n \longrightarrow \mathbb{K}$ which fails to be $(p_1, \ldots, p_n)$-dominated.*

Acknowledgement. The authors thank Joe Diestel for helpful conversations on the subject of this paper.

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